Robust Optimization for Unit Commitment and Dispatch in Energy Markets

Marco Zugno*, Juan Miguel Morales, Henrik Madsen (DTU Compute)
and Antonio Conejo (Ohio State University)

* email: mazu@dtu.dk

Modeling and Optimization of Heat and Power System — DTU, 12th January 2015

Outline

Why Optimization Under Uncertainty

Decision-making in energy markets

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Sources of uncertainty in energy systems

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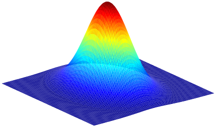


Better decisions if optimization accounts for uncertainty

Deterministic Optimization Framework

Unit commitment and dispatch formalized as optimization problems

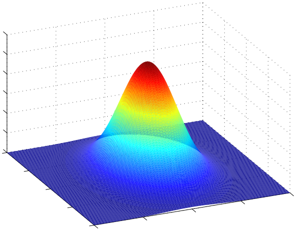
$$\begin{aligned} \underset{\mathbf{x},\mathbf{y}}{\text{Min.}} \ & \mathbf{c}^\top \mathbf{x} + \mathbf{q}^\top \mathbf{y} \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} \geq \mathbf{b} \ , \\ & \mathbf{T}\mathbf{x} + \mathbf{W}\mathbf{y} \geq \mathbf{h} \end{aligned}$$

- $\mathbf{x}, \mathbf{y}$ represent decision variables: unit status, dispatch, storage level, network state, etc.
- inequality and equality constraints impose limits on dispatch: dispatch limits, ramping, supply/demand balance, etc.

Optimization Under Uncertainty

Most often not all parameters are known in advance

$$\begin{aligned} \underset{\mathbf{x},\mathbf{y}_\omega}{\text{Min.}}\ & \mathbf{c}^\top \mathbf{x} + \mathcal{M}_\omega \left\{ \mathbf{q}_\omega^\top \mathbf{y}_\omega \right\} \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} \geq \mathbf{b} \ , \\ & \mathbf{T}\mathbf{x} + \mathbf{W}\mathbf{y}_\omega \geq \mathbf{h}_\omega \ , \qquad \forall \omega \in \Omega \end{aligned}$$



- Uncertain parameters depend on random variable ω
- Variables $\mathbf{y}_\omega$ are adjustable (recourse)
- Measure of recourse cost in the objective function (expectation, etc.)
- Constraints hold for different realizations of the uncertainty

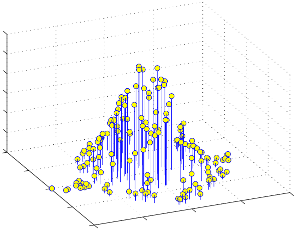
Stochastic Programming

Tractable version of stochastic problem by sampling the uncertainty

$$\underset{\mathbf{x},\mathbf{y}_{\omega_s}}{\text{Min.}}\ \mathbf{c}^\top \mathbf{x} + \sum_{s=1}^{S} p_{\omega_s} \mathbf{q}_{\omega_s}^\top \mathbf{y}_{\omega_s}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} \geq \mathbf{b}\ ,$$

$$\mathbf{T}\mathbf{x} + \mathbf{W}\mathbf{y}_{\omega_s} \geq \mathbf{h}_{\omega_s}\ , \qquad s = 1, \ldots, S$$

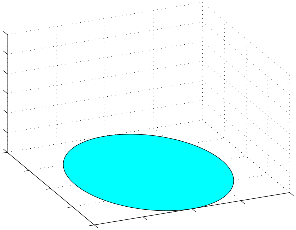


- Probably the most popular method of optimization under uncertainty
- Only used as a comparison in this presentation

Adaptive Robust Optimization (ARO)

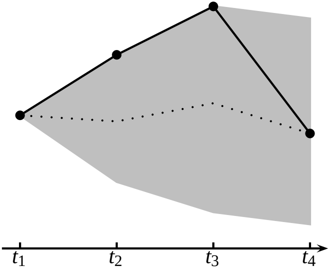
Tractable version of stochastic problem based on notion of uncertainty set

$$\begin{aligned} \underset{\mathbf{x}}{\mathrm{Min.}}\ & \mathbf{c}^\top \mathbf{x} + \max_{\mathbf{q},\mathbf{h}} \min_{\mathbf{y}} \mathbf{q}^\top \mathbf{y} \\ & \qquad\qquad \mathbf{Tx} + \mathbf{Wy} \geq \mathbf{h}\ , \\ & \qquad (\mathbf{q}, \mathbf{h}) \in \mathcal{U}\ , \\ \mathrm{s.t.}\ & \mathbf{Ax} \geq \mathbf{b}\ , \end{aligned}$$



- $\mathcal{M}(\cdot)$ replaced by worst-case realization of recourse cost
- Inner minimization problem guarantees feasibility of solution
- Three-level game against nature

ARO and Non-Anticipativity



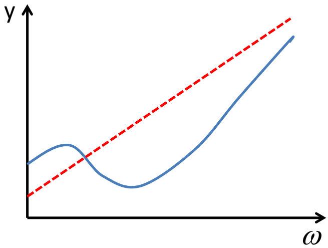
- Worst-case trajectory of uncertainty revealed **before** making recourse decisions
- Recourse has perfect knowledge of future worst-case uncertainty

Affinely Adjustable Robust Optimization (AARO)

Enforcing Linear Decision Rules

Optimal recourse functions $\mathbf{y}(\omega)$ approximated by linear rules:

$$\mathbf{y} = \mathbf{Y}\omega$$



Computational Simplification

Linear coefficient $\mathbf{Y}$ is a first-stage variable: essentially a max-min problem

Affinely Adjustable Robust Optimization (AARO)

Problem Formulation

$$\begin{aligned} \underset{\mathbf{x},\mathbf{Y}}{\text{Min.}}\ & \mathbf{c}^\top \mathbf{x} + \mathbb{E}_\omega \left\{ \omega^\top \mathbf{Q}^\top \mathbf{Y} \omega \right\} \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} \geq \mathbf{b}\ , \\ & \mathbf{T}\mathbf{x} + \mathbf{W}\mathbf{Y}\omega \geq \mathbf{H}\omega\ , \quad \forall \omega \in \mathcal{U} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \underset{\mathbf{x},\mathbf{Y}}{\text{Min.}}\ & \mathbf{c}^\top \mathbf{x} + \mathbb{E}_\omega \left\{ \omega^\top \mathbf{Q}^\top \mathbf{Y} \omega \right\} \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} \geq \mathbf{b}\ , \\ & \max_{\omega \in \Omega} \left\{ \mathbf{T}\mathbf{x} + (\mathbf{W}\mathbf{Y} - \mathbf{H})\,\omega \right\} \geq 0 \end{aligned}$$

Differences from ARO:

- Not bound to optimize worst-case cost: can be made less conservative
- min-max problem can be simplified via duality argument: single stage problem

Energy and Reserve Dispatch using Adaptive Robust Optimization

Electricity Market Setup

- Day-ahead market: 12- to 36-hour ahead
- Energy and reserve are co-optimized (first stage)
- Followed by a balancing (real-time) market (second stage)
- Network is represented—as in US markets
- The right of starting-up and shutting-down own units is not confiscated—as in EU markets

Adaptive Robust Optimization Approach

Two-stage robust optimization turn into three-level optimization problems

1. Make dispatch decision $\mathbf{x}$ with prognosis of the future (minimize cost)
2. Uncertainty (wind power production $\Delta\mathbf{w}$) unfolds (maximize cost)
3. Make operation (recourse) decision $\mathbf{y}$ (minimize cost)

The Robust Optimization Approach

$$
\begin{aligned}
\min_{\mathbf{x}} \;\; & \mathbf{c_x}^T \mathbf{x} + \max_{\Delta \mathbf{w}} \; \min_{\mathbf{y}} \;\; \mathbf{c_y}^T \mathbf{y} \\
& \qquad\qquad\qquad \text{s.t.} \;\; \mathbf{P y} = -\Delta \mathbf{w} - \mathbf{Q x} \, , \qquad\qquad : \boldsymbol{\lambda} \, , \\
& \qquad\qquad\qquad\qquad\;\; \mathbf{L y} \geq \mathbf{l} - \mathbf{M x} - \mathbf{N} \Delta w \, , \qquad : \boldsymbol{\mu} \, , \\
& \qquad\qquad \text{s.t.} \;\; \mathbf{H} \Delta w \leq \mathbf{h} \, , \\
\text{s.t.} \;\; & \mathbf{F x} = \mathbf{d} - \widehat{\mathbf{w}} \, , \\
& \mathbf{G x} \geq \mathbf{g} \, .
\end{aligned}
$$

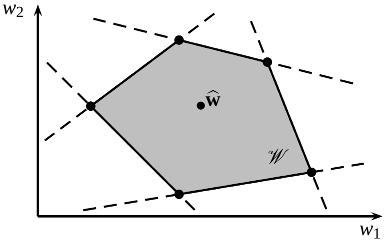
This is a complicated problem!

There is no general solution technique for three-level optimization problems

Modeling the Uncertainty Set

Uncertain stochastic power production at different nodes described by polyhedral uncertainty set $\mathcal{W}$

- can be modeled via linear inequalities $\mathbf{H}\boldsymbol{\Delta}\mathbf{w} \leq \mathbf{h}$
- can include "budget constraints" on the stochastic production
- can limit the variation of stochastic production among adjacent plants



Column & Constraint Generation Approach

In the linear case, for any feasible $\mathbf{x}$

- the worst-case realization of $\mathbf{w}$ is a vertex of the polytope $\mathcal{W}$
- the worst-case recourse cost is the maximum of a finite number of affine functions of x

$$\Downarrow$$

1. Find optimal first-stage solution for approximation (lower bound)
2. Calculate worst-case recourse cost (upper bound)
3. Generate cut

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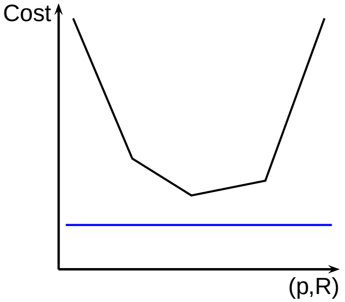

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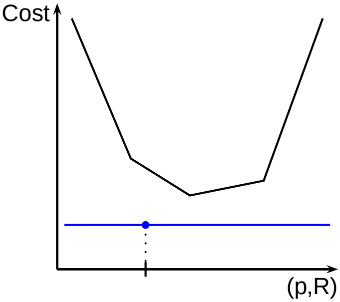

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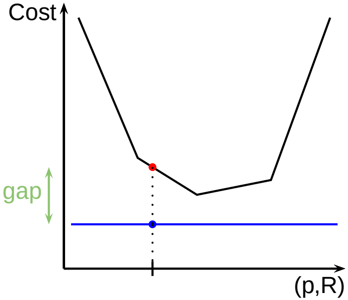

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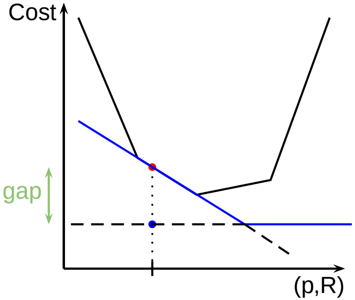

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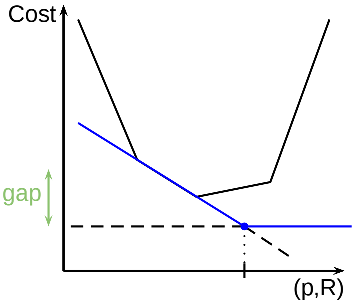

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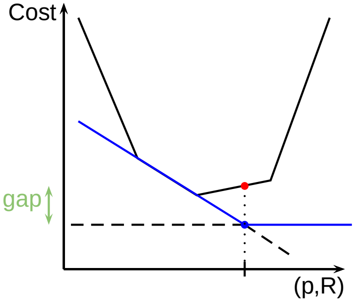

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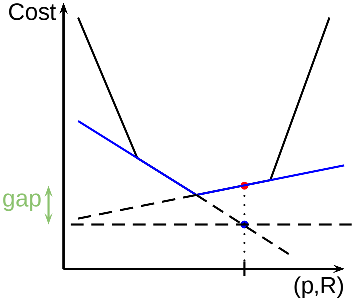

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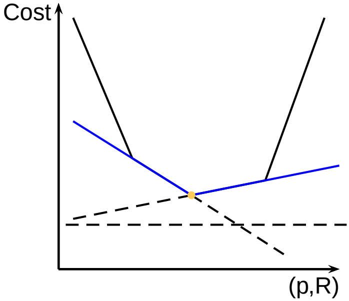
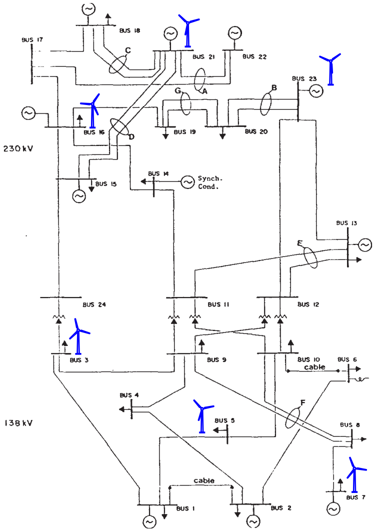


Modified IEEE 24-Node Reliability Test System

- Block offers by generators
- Flexible generators provide reserve
- Three lines with reduced capacity
- Wind farms at 6 nodes of the system



Wind Power Data

- Data from 6 locations in Spain
- Uncertainty set parameters estimated from historical data
- Historical data also used as scenarios

Uncertainty Set for Wind Power Production

- Budget of uncertainty:
  - Intervals: $-\Delta W_n^{\max} \leq \Delta w_n \leq \Delta W_n^{\max}$, $\qquad \forall n \in \Phi^N$
  - Budget: $\sum_{n \in \Phi^N} \frac{|\Delta w_n|}{\Delta W_n^{\max}} \leq \Gamma$

- One additional type of constraint to model geographical correlation

$$-\rho_{n_1 n_2} \leq \frac{\Delta w_{n_1}}{\Delta w_{n_1}^{\max}} - \frac{\Delta w_{n_2}}{\Delta w_{n_2}^{\max}} \leq \rho_{n_1 n_2} \ , \qquad \forall n_1, n_2 \in \Phi^N, n_1 \neq n_2$$

Robust Optimization vs Stochastic Programming

Risk-Neutral Case

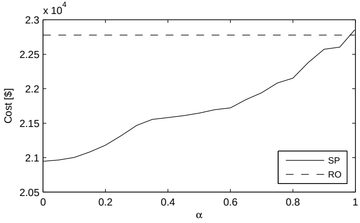
Cost	Robust Optimization			Stochastic Programming		
Energy disp.	19 015.55			17 525.95		
Up reserve	**1529.04**			**1395.77**		
Down reserve	**0**			**526.68**		
Total day-ahead	20 544.59			19 448.40		
	Mean	$\text{CVaR}_{95\%}$	$\text{VaR}_{100\%}$	Mean	$\text{CVaR}_{95\%}$	$\text{VaR}_{100\%}$
Energy redisp.	2671.12	7223.69	7943.22	2176.31	9236.00	10 045.83
Load-shedding	19.49	374.75	9743.38	39.06	751.15	19 529.87
Total balancing	2690.60	7598.44	17 686.60	2215.37	9987.15	29 575.70
Total aggregate	**23 235.19**	28 143.03	**38 231.19**	**21 663.77**	29 435.55	**49 024.10**

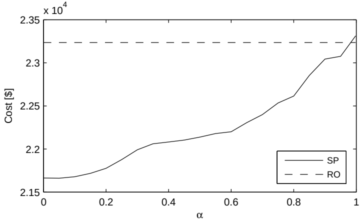
Robust Optimization vs Stochastic Programming

Risk-Averse Case

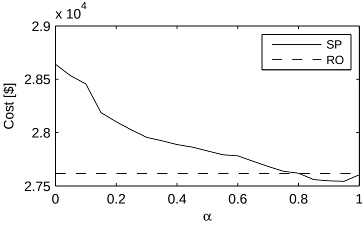
In-sample | Out-of-sample

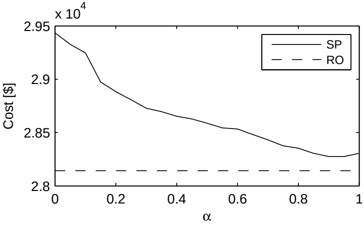
Expectation





CVaR

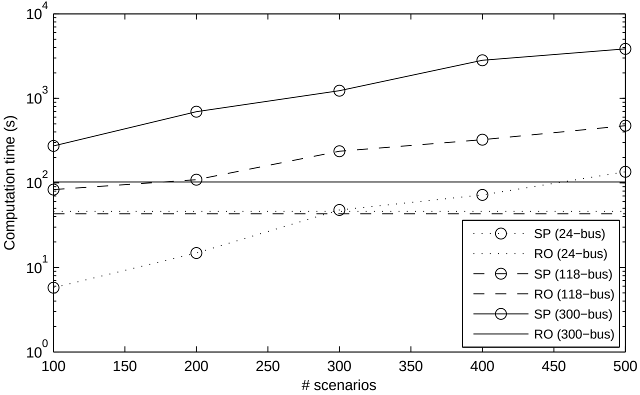




Robust Optimization vs Stochastic Programming

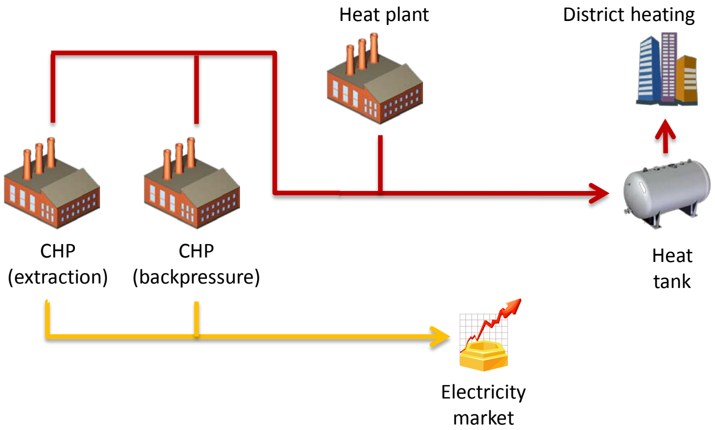
Computational Properties

Computational results with different IEEE power system models



Unit Commitment and Scheduling for Heat & Power Systems using Affinely Adjustable Robust Optimization

Heat and Power System Example



A Multi-Stage Uncertain Problem

The problem is uncertain (heat demand, prices) and multi-stage:

Day-ahead Determine unit commitment and schedule for the following day (periods $1, \ldots, N_T$)

Oper. h 1 Uncertainty at time 1 unfolds and units are re-dispatched (based on realizations until time 1)

Oper. h 2 Uncertainty at time 2 unfolds and units are re-dispatched (based on realizations until time 2)

⋮

Oper. h N_T Uncertainty at time N_T unfolds and units are re-dispatched (based on all realizations)

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Robust Optimization Approach

Our approach based on Robust Optimization (RO)

$$\underset{\mathbf{x},\mathbf{y}(\cdot)}{\text{Minimize}}\ \mathbf{c}^\top \mathbf{x} + \mathbb{E}_\omega \left\{ \mathbf{q}_\omega^\top \mathbf{y}(\omega) \right\} \tag{1}$$

$$\text{s.t. } \mathbf{A}\mathbf{x} \geq \mathbf{b}\ , \tag{2}$$

$$\mathbf{T}\mathbf{x} + \mathbf{W}\mathbf{y}(\omega) \geq \mathbf{h}_\omega\ , \qquad \forall \omega \in \mathcal{U} \tag{3}$$

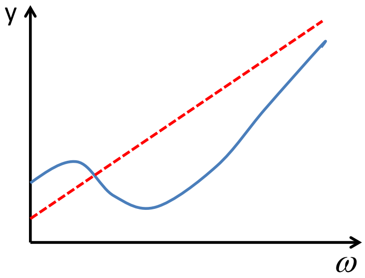
where

- $\mathbf{x}$ are first-stage decision (unit commitment, schedule)
- $\mathbf{y}(\cdot)$ are **recourse functions** of the uncertainty ω (unit operation)
- The solution is optimal in expectation (1)
- The solution is feasible for any ω in an **uncertainty set** $\mathcal{U}$ (3)

Linear Decision Rules

Optimal recourse decision approximated via linear decision rules

$$y_{it}(\omega) = \widehat{y}_{it} + \sum_{\tau=1}^{t} Y_{it\tau}\omega_\tau \ , \qquad \forall i, t$$



- Recourse decision depends on history of uncertainty—up to now (t)
- Independence on future uncertainty realizations ($\tau > t$): non-anticipative

Final Linear Reformulation

We make the following assumptions:

- linear dependence of parameters to uncertainty: $\mathbf{q}_\omega = \mathbf{Q}\omega$, $\mathbf{h}_\omega = \mathbf{H}\omega$
- budget uncertainty set defined by $\mathbf{D}\omega \geq \mathbf{d}$

By duality arguments, the problem can be reformulated linearly:

$$\begin{aligned}
\underset{\mathbf{x},\mathbf{Y},\boldsymbol{\Lambda}}{\text{Min.}}\ & \mathbf{c}^\top \mathbf{x} + \text{tr}\left\{ \mathbf{Q}^\top \mathbf{Y} \left(\boldsymbol{\Sigma}_\omega + \mathbb{E}\{\omega\}\mathbb{E}\{\omega\}^\top \right) \right\} \\
\text{s.t.}\ & \mathbf{A}\mathbf{x} \geq \mathbf{b}\ , \\
& \boldsymbol{\Lambda}^\top \mathbf{d} \geq -\mathbf{T}\mathbf{x}\ , \\
& \mathbf{D}^\top \boldsymbol{\Lambda} = (\mathbf{W}\mathbf{Y} - \mathbf{H})^\top\ , \\
& \boldsymbol{\Lambda} \in \mathbb{R}_{\geq 0}^{R \times L_2}
\end{aligned}$$

Case-Study Setup

Three production facilities with storage

- Extraction CHP (AMV3, AVV1)
- Back-pressure CHP (AMV1)
- heat-only plant

Historical heat load from VEKS

- RO: budget uncertainty set (tunable interval size and budget)
- SP: Gaussian scenarios

Power prices from NordPool

- SP: Gaussian scenarios correlated with heat demand

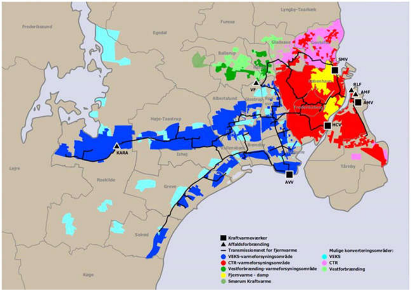
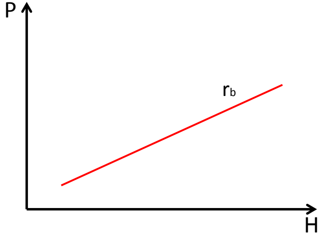


Figure from HOFOR.dk

Back-pressure vs Extraction CHP Units

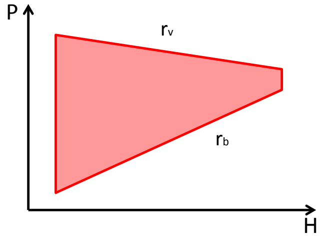
Back-pressure CHP unit

- constant heat/power ratio (r_b)



Extraction CHP unit

- more flexible operation in heat/power space
- feasible region approximated by polyhedron

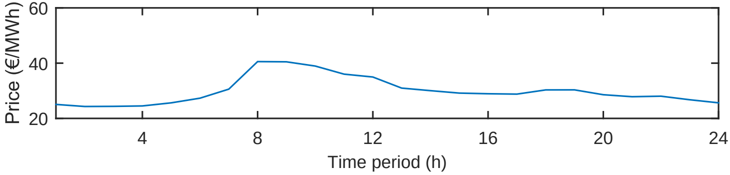


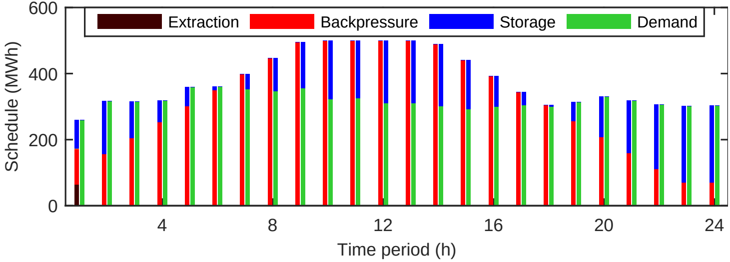
Parameters of the System

		Extraction	Back-pressure	Heat-only
Power/heat ratio	r_b	0.64	0.28	0
Electricity loss for heat prod.	r_v	-0.13	–	–
Fuel per electricity unit	φ^p	2.40	2.40	–
Fuel per heat unit	φ^q	0.31	0.36	1.09
Minimum fuel input	$\underline{f}$	120	72.24	0
Maximum fuel input	$\overline{f}$	631.20	516	1086.96
Ramp-up/-down for fuel input	$\overline{r}/-\underline{r}$	150	50	1086.96
Minimum heat output	$\underline{q}$	0	70	0
Maximum heat output	$\overline{q}$	331	500	1000
Fuel marginal cost	c	24.16	12.75	93.96
No-load cost	c^0	0	0	2684.56
Start-up/shut-down cost	$c^{\mathrm{SU}}/c^{\mathrm{SD}}$	7 382.55	6 040.27	0
Minimum up-time	T^{U}	2	5	0
Minimum down-time	T^{D}	2	5	0
Provides real-time flexibility		yes	no	yes

Scheduling Example

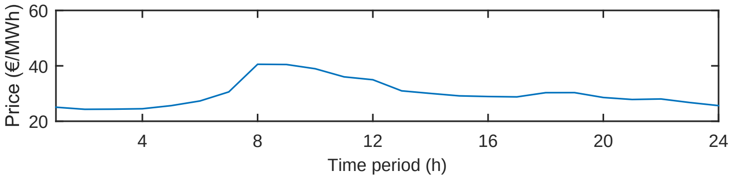
Results from the Deterministic Model

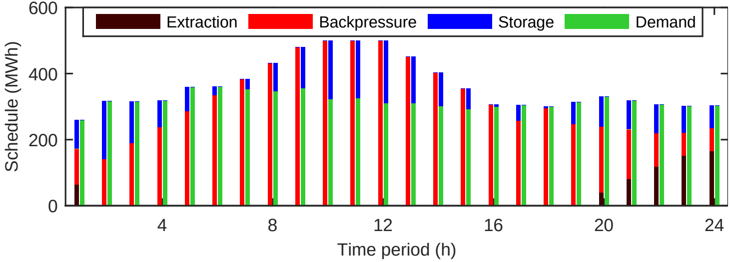




Scheduling Example

Results from the Robust Optimization Model





Tuning the Need for Flexibility

The extraction unit is the main provider of flexibility

	Unit	Γ				
		2	**4**	**6**	**8**	**10**
Total heat dispatch	MWh	265.20	444.27	622.24	811.31	1250.74
Periods online	h	4	5	6	8	24

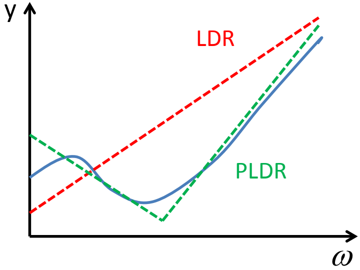
Piecewise Linear Decision Rules

We can define different decision rules for partitions of the uncertainty set

- 2 different rules for each dimension of the uncertainty
- partitioning demand deviation about 0 straightforward choice

$$\Downarrow$$

different response to deficit/surplus heat demand



Linear vs Piecewise Linear Decision Rules

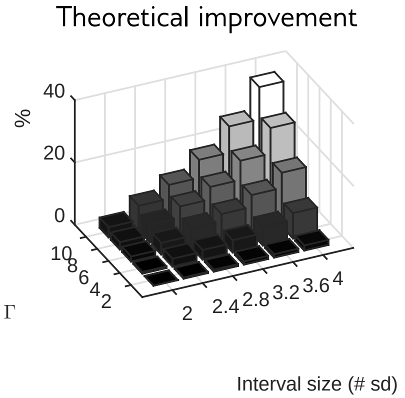
Dispatch Improvement

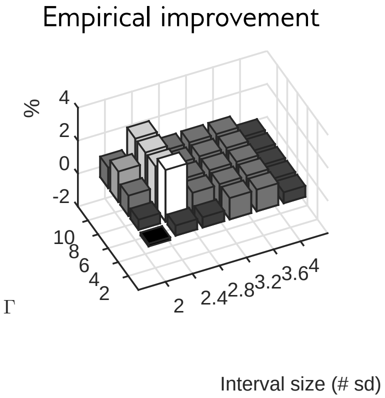
Dispatch for extraction unit during peak-demand day

Decision rule	**Unit**	**Γ**				
		2	**4**	**6**	**8**	**10**
Linear	MWh	7944.00	7783.77	7572.89	7377.16	7200.06
Piecewise-linear	MWh	7944.00	7944.00	7944.00	7944.00	7944.00

Linear vs Piecewise Linear Decision Rules

In-Sample vs Out-of-Sample Results

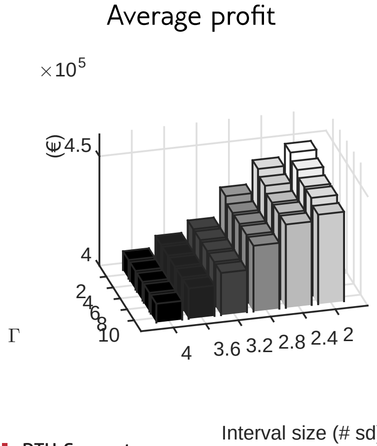


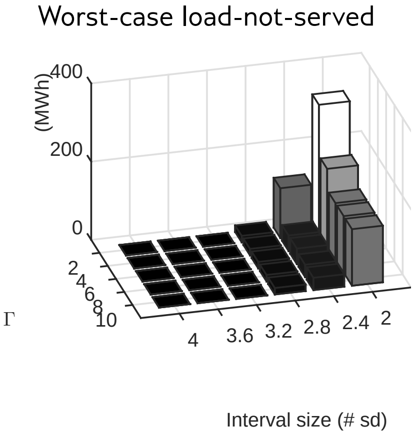


- decision rules not binding in practice
- anticipative out-of-sample evaluation (stochastic programming)

Price of Robustness

Sensitivity analysis can help tune the RO parameters (cost vs robustness)





RO vs Alternative Methods

Out-of-Sample Analysis

Model	Interval radius (# sd)	Γ	Average profit (€)	Load not served (MWh)	
				largest	expected
	2.00	2	449 301.37	329.38	7.88
	2.00	4	446 337.45	198.93	3.38
	2.00	6	443 598.68	144.68	2.51
RO-PLDR	2.40	2	443 474.44	127.20	0.51
	2.40	4	440 776.45	39.64	0.29
	2.40	8, 10	438 086.22	31.58	0.20
	2.80	2	432 707.89	16.59	0.05
	3.20	2–10	419 281.79	0.00	0.00
DET	–	–	455 815.78	745.78	69.52
SP	–	–	448 945.73	159.90	2.13

- Load-not-served not penalized in the objective function
- Out-of-sample analysis based on SP

Conclusion

- Robust optimization addresses some of the shortcomings of stochastic programming
  - intractability (large scenario sets)
  - violation of non-anticipativity in multi-stage problems
- Simpler models of the uncertainty are required
  - uncertainty sets
  - low-order moments (mean, standard deviation, etc.)
- Trade-off between optimality and conservativeness can be tuned via uncertainty set parameters
  - similar results in terms of CVaR for risk-averse SP (ARO model for energy and reserve)
  - tunable conservativeness in AARO model for CHP units
- Promising results in terms of scalability

References

- M. Zugno, A.J. Conejo (2014). *A Robust Optimization Approach to Energy and Reserve Dispatch in Electricity Markets,* (under review).
- M. Zugno, J.M. Morales, H. Madsen (2014). *Commitment and Dispatch of Heat and Power Units via Affinely Adjustable Robust Optimization* (working paper, available soon).
- M. Zugno, J.M. Morales, H. Madsen (2014). *Robust Management of Combined Heat and Power Systems via Linear Decision Rules,* 2014 IEEE Energy Conference (ENERGYCON) Proceedings, 479–486.

Thank you for your attention!

Q&A Session

Inner Problem Reformulation via Duality

From *min-max-min* to *min-max-max*:

$$\min_{\mathbf{x}} \mathbf{c_x}^T\mathbf{x} + \max_{\Delta \mathbf{w}} \max_{\boldsymbol{\lambda},\boldsymbol{\mu}} \ (-\Delta w - \mathbf{Q}\mathbf{x})^T \boldsymbol{\lambda} + (\mathbf{I} - \mathbf{M}\mathbf{x} - \mathbf{N}\Delta w)^T \boldsymbol{\mu}$$

$$\text{s.t. } \mathbf{P}^T\boldsymbol{\lambda} + \mathbf{L}^T\boldsymbol{\mu} = \mathbf{c_y} \ ,$$

$$\boldsymbol{\mu} \geq 0 \ ,$$

$$\text{s.t. } \mathbf{H}\Delta w \leq \mathbf{h} \ ,$$

$$\text{s.t. } \mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}} \ ,$$

$$\mathbf{G}\mathbf{x} \geq \mathbf{g} \ .$$

Inner *max-max* problem can be merged into a single **bilinear** maximization problem

Solution via Cutting-Plane Method

$$\min_{\mathbf{x},\beta} \; {\mathbf{c_x}}^T \mathbf{x} + \beta$$

$$\text{s.t.} \;\; \beta \geq -\Delta \mathbf{w}_k^T \boldsymbol{\lambda}_k + \left(\mathbf{I} - \mathbf{N}\Delta \mathbf{w}_k\right)^T \boldsymbol{\mu}_k - \left(\boldsymbol{\lambda}_k \mathbf{Q} + \boldsymbol{\mu}_k \mathbf{M}\right) \mathbf{x} \; , \qquad \forall k \; ,$$

$$\mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}} \; ,$$

$$\mathbf{G}\mathbf{x} \geq \mathbf{g} \; .$$

1. Start with a lower bound β for the recourse cost
2. Find optimal first-stage solution x for the approximation (lower bound)
3. Calculate worst-case recourse cost (upper bound)
4. Generate cut for the vertex, go back to 2

Solution via Cutting-Plane Method

$$
\begin{aligned}
\min_{\mathbf{x},\beta} \quad & \mathbf{c_x}^T \mathbf{x} + \beta \\
\text{s.t.} \quad & \beta \geq -\Delta \mathbf{w}_k^T \boldsymbol{\lambda}_k + (\mathbf{I} - \mathbf{N}\Delta \mathbf{w}_k)^T \boldsymbol{\mu}_k - (\boldsymbol{\lambda}_k \mathbf{Q} + \boldsymbol{\mu}_k \mathbf{M})\,\mathbf{x}\,, \qquad \forall k\,, \\
& \mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}}\,, \\
& \mathbf{G}\mathbf{x} \geq \mathbf{g}\,.
\end{aligned}
$$

While (upper bound - lower bound > ϵ)

1. Start with a lower bound β for the recourse cost
2. Find optimal first-stage solution x for the approximation (lower bound)
3. Calculate worst-case recourse cost (upper bound)
4. Generate cut for the vertex, go back to 2

Solution via Cutting-Plane Method

$$\min_{\mathbf{x},\beta} \; {\mathbf{c_x}}^T \mathbf{x} + \beta$$

$$\text{s.t.} \;\; \beta \geq -\Delta \mathbf{w}_k^T \boldsymbol{\lambda}_k + (\mathbf{I} - \mathbf{N}\Delta\mathbf{w}_k)^T \boldsymbol{\mu}_k - (\boldsymbol{\lambda}_k \mathbf{Q} + \boldsymbol{\mu}_k \mathbf{M})\,\mathbf{x} \,, \qquad \forall k \,,$$

$$\mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}} \,,$$

$$\mathbf{G}\mathbf{x} \geq \mathbf{g} \,.$$

While (upper bound - lower bound > ϵ)

1. Start with a lower bound β for the recourse cost
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Solution via Cutting-Plane Method

$$\min_{\mathbf{x},\beta} \; \mathbf{c_x}^T \mathbf{x} + \beta$$
$$\text{s.t.} \; \beta \geq -\Delta\mathbf{w}_k^T \boldsymbol{\lambda}_k + (\mathbf{I} - \mathbf{N}\Delta\mathbf{w}_k)^T \boldsymbol{\mu}_k - (\boldsymbol{\lambda}_k \mathbf{Q} + \boldsymbol{\mu}_k \mathbf{M})\,\mathbf{x} \,, \qquad \forall k \,,$$
$$\mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}} \,,$$
$$\mathbf{G}\mathbf{x} \geq \mathbf{g} \,.$$

While (upper bound - lower bound > ϵ)

1. Start with a lower bound β for the recourse cost
2. Find optimal first-stage solution x for the approximation (lower bound)
3. Calculate worst-case recourse cost (upper bound)
4. Generate cut for the vertex, go back to 2

Solution via Cutting-Plane Method

$$\min_{\mathbf{x},\beta} \quad \mathbf{c_x}^T\mathbf{x} + \beta$$

$$\text{s.t.} \quad \beta \geq -\Delta\mathbf{w}_k^T\boldsymbol{\lambda}_k + (\mathbf{I} - \mathbf{N}\Delta\mathbf{w}_k)^T\boldsymbol{\mu}_k - (\boldsymbol{\lambda}_k\mathbf{Q} + \boldsymbol{\mu}_k\mathbf{M})\,\mathbf{x}\,, \qquad \forall k\,,$$

$$\mathbf{F}\mathbf{x} = \mathbf{d} - \widehat{\mathbf{w}}\,,$$

$$\mathbf{G}\mathbf{x} \geq \mathbf{g}\,.$$

While (upper bound - lower bound > ϵ)

1. Start with a lower bound β for the recourse cost
2. Find optimal first-stage solution x for the approximation (lower bound)
3. Calculate worst-case recourse cost (upper bound)
4. Generate cut for the vertex, go back to 2

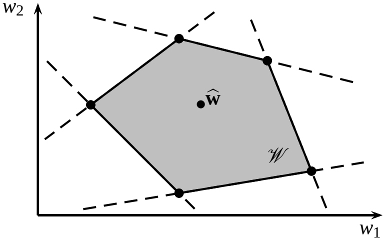
Subproblem Choice vs Uncertainty Set

Point 3 in the cutting-plane method requires the solution of a bilinear program

- Simple, "easy-to-enumerate" uncertainty sets (budget)
- Heuristic methods (suboptimal)

We want to be able to solve the problem

- **exactly**
- for **any polyhedral** uncertainty set



So far only "easy-to-enumerate" polyhedral sets have been considered, or heuristic techniques

Reformulation of the Problem (1)

Swapping Inner Problems

The inner problems can be swapped:

$$
\begin{aligned}
\min_{\mathbf{x}}\ & \mathbf{c_x}^T\mathbf{x} + \max_{\boldsymbol{\lambda},\boldsymbol{\mu}}\ -(\mathbf{Qx})^T\boldsymbol{\lambda} + (\mathbf{l}-\mathbf{Mx})^T\boldsymbol{\mu} + \max_{\Delta\mathbf{w}}\ -\left(\boldsymbol{\lambda}^T + \boldsymbol{\mu}^T\mathbf{N}\right)\Delta\mathbf{w} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{s.t. } \mathbf{H}\Delta\mathbf{w} \le \mathbf{h}\ , \quad :\boldsymbol{\xi}\ , \\
& \qquad\qquad \text{s.t. } \mathbf{P}^T\boldsymbol{\lambda} + \mathbf{L}^T\boldsymbol{\mu} = \mathbf{c_y}\ , \\
& \qquad\qquad\qquad \boldsymbol{\mu} \ge 0\ , \\
\text{s.t. } & \mathbf{Fx} = \mathbf{d} - \widehat{\mathbf{w}}\ , \\
& \mathbf{Gx} \ge \mathbf{g}\ ,
\end{aligned}
$$

Reformulation of the Problem (2)

Using Strong Duality

The inner problem is linear $\Rightarrow$ strong duality holds:

$$-\underbrace{\left(\boldsymbol{\lambda}^T + \boldsymbol{\mu}^T \mathbf{N}\right) \Delta \mathbf{w}}_{\text{objective function}} = \mathbf{h}^T \boldsymbol{\xi}$$

We can exchange a bilinear term with a linear one, but the innermost problem becomes a minimization problem

- We can enforce KKT conditions to guarantee optimality and get rid of a level

Reformulation of the Problem (3)

Final Formulation

$$\min_{\mathbf{x}} \mathbf{c_x}^T\mathbf{x} + \max_{\boldsymbol{\lambda},\boldsymbol{\mu},\Delta\mathbf{w},\boldsymbol{\xi}} -(\mathbf{Qx})^T\boldsymbol{\lambda} + (\mathbf{l}-\mathbf{Mx})^T\boldsymbol{\mu} + \mathbf{h}^T\boldsymbol{\xi}$$

$$\text{s.t. } \mathbf{0} \leq \boldsymbol{\xi} \perp \mathbf{h} - \mathbf{H}\Delta\mathbf{w} \geq \mathbf{0} ,$$

$$\mathbf{H}^T\boldsymbol{\xi} = -\boldsymbol{\lambda} - \mathbf{N}^T\boldsymbol{\mu} ,$$

$$\mathbf{P}^T\boldsymbol{\lambda} + \mathbf{L}^T\boldsymbol{\mu} = \mathbf{c_y} ,$$

$$\boldsymbol{\mu} \geq \mathbf{0} ,$$

$$\text{s.t. } \mathbf{Fx} = \mathbf{d} - \widehat{\mathbf{w}} ,$$

$$\mathbf{Gx} \geq \mathbf{g} .$$

$\perp$ indicates that the product of the operands is 0 $\Rightarrow$ this can be linearized using binary variables (MILP)

RO vs Alternative Methods

Out-of-Sample Analysis (Spring)

Model	Interval radius (# sd)	Γ	Average profit (€)	Heat load not served (MWh)	
				largest	expected
	2.00	2	251 514.22	217.26	5.70
RO-PLDR	2.40	2	250 505.55	6.78	0.01
	4.00	2	248 484.94	0.00	0.00
DET	–	–	260 927.88	340.15	49.74
SP	–	–	257 295.35	76.17	1.54

RO vs Alternative Methods

Out-of-Sample Analysis (Summer)

Model	Interval radius (# sd)	Γ	Average profit (€)	Heat load not served (MWh)	
				largest	expected
RO-PLDR	2.00	2	88 423.27	0.00	0.00
DET	–	–	94 413.41	117.00	16.10
SP	–	–	92 356.87	0.00	0.00

RO vs Alternative Methods

Out-of-Sample Analysis (Autumn)

Model	Interval radius (# sd)	Γ	Average profit (€)	Heat load not served (MWh)	
				largest	expected
RO-PLDR	2.00	2	249 631.81	90.35	0.41
	2.40	2	248 673.51	0.00	0.00
DET	–	–	259 135.23	238.72	12.01
SP	–	–	255 836.17	99.71	0.54